

Geodesics in Open Universes

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We present the geodesics on homogeneous and isotropic negatively curved spaces in a simple form suitable for application to cosmological problems. We discuss how the patterns in the microwave sky of anisotropic homogeneous universes can be predicted qualitatively by looking at the invariances that generate their three-geometries.

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Metric theories of gravity, of which Einstein's general theory of relativity is the most elegant known example, couple the material content of the Universe to its space-time geometry. As a result the paths of light rays reflect the curvature of space and provide a means to determine the density and overall curvature of the Universe. Locally, the effect of the Sun's gravitational field upon grazing light rays and radio waves provides a powerful test of Einstein's predictions. Globally, we have begun to observe the effects of gravitational lensing, and we know that varying the density of the Universe will alter the observed features of microwave background patterns traced by photons on the sky. Although most versions of the inflationary universe scenario lead us to expect that the universe will be expanding very close to the critical divide, with the density parameter satisfying $|\Omega_0 - 1| \sim O(10^{-5})$, there are varieties of inflation which predict that the universe is significantly open ($\Omega_0 \leq O(0.1)$) [1]. Moreover, the observational evidence stubbornly refuses to provide a clear endorsement of the $\Omega_0 \sim 1$ predictions. Primordial nucleosynthesis limits the baryon density to fall well short of this value, and so non-baryonic forms of dark matter must be found in support of a closure density. For these reasons, the determination of all the observational differences between flat, closed, and open universes is an important goal for cosmologists. The most sensitive discriminators promise to come from the study of null geodesics.

There has been much recent interest in studying the observational signatures of open universes that possess compact topologies which are produced by the periodic identification of space coordinates, and of non-compact topologies that possess identifications in some space directions [2,3,4] which permits them to have integrable geodesic motions. Open universes are interesting candidates for possessing non-trivial topologies because the curvature of space provides a natural length scale to relate to the scale of topological identification. There have been several investigations of the effects of topological identifications on the power spectrum of the microwave

sky and on simulated COBE sky maps. These studies enable us to limit the scale of topological identifications more powerfully than by searching for multiple images of prominent luminous sources [2]. In these studies, the negative curvature of open universes plays an important role. Geodesic flows on compact negatively-curved spaces are chaotic, as first noted by Hadamard [5], and the geodesic flow on a compact negatively-curved space has become a key paradigm for the identification of chaotic classical motions as well as in the study of quantum chaos. Some possible implications for the microwave background have been discussed in ref. [6]. However, the formal characterisation of these mixing flows [7] is of little practical utility for the study of geodesics in the universe. The needs of cosmologists are more specific and in this paper we aim to provide the geodesics for open universes in simple usable form. These do not seem to exist in the literature and can be employed to study the behaviour of the cosmic background radiation in open universes with any topology. With such applications in mind, we provide a brief analysis of the geodesics on an expanding negatively-curved space. They are presented in an immediately accessible form for use in cosmological investigations. Finally, we provide some new qualitative discussion of the microwave sky patterns in anisotropic universes.

II. GEODESICS ON A STATIC, NEGATIVELY CURVED SPACE

The most familiar description of the metric in an open universe is in hyperbolic coordinates

$$ds^2 = -dt^2 + dr^2 + \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2) \quad . \quad (1)$$

However, the geodesics can be found more simply in a coordinate system (x, y, x) related to (r, θ, ϕ) by

$$\begin{aligned} e^{-z} &= \cosh r - \sinh r \cos \theta \\ e^{-z} x &= \sin \theta \cos \phi \sinh r \\ e^{-z} y &= \sin \theta \sin \phi \sinh r \quad . \end{aligned} \quad (2)$$

In the (x, y, z) coordinate system the metric is

$$ds^2 = -dt^2 + dz^2 + e^{-2z}(dx^2 + dy^2) . \quad (3)$$

The geodesic equations can be found in the usual way from

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dz^\beta}{d\lambda} = 0 . \quad (4)$$

but it is more efficient to introduce the Lagrangian

$$L = \frac{1}{2} [-t'^2 + z'^2 + e^{-2z}(x'^2 + y'^2)] \quad (5)$$

where $' = d/d\lambda$. The equations of motion can be found from

$$\Pi'_q - \frac{\partial L}{\partial q} = 0 \quad (6)$$

where $\Pi_q = \partial L / \partial q'$ is the momentum conjugate to coordinate q .

Since the Lagrangian is independent of (t, x, y) , the corresponding conjugate momenta are conserved, giving

$$\begin{aligned} \Pi_t &= -t' = -E_i \\ \Pi_x &= e^{-2z}x' = \Pi_{xi} \\ \Pi_y &= e^{-2z}y' = \Pi_{yi} \end{aligned} \quad (7)$$

where E_i, Π_{xi}, Π_{yi} are constants of the motion. The last of these equations is

$$z'' + e^{-2z}(x'^2 + y'^2) = 0 . \quad (8)$$

This second-order equation can be reduced to first order by exploiting the invariance of the length element. Choosing the affine parameter to be proper time η , $ds^2/d\eta^2 = \alpha$ ($\alpha = 0$ for photons and $\alpha = -1$ for massive particles), we have (3)

$$\alpha = -E_i^2 + z'^2 + e^{-2z}(x'^2 + y'^2) . \quad (9)$$

Substituting the solutions for x' and y' and defining $W_i^2 = \Pi_{xi}^2 + \Pi_{yi}^2$, we can solve this for z' to find

$$z' = \pm [(\alpha + E_i^2) - e^{2z}W_i^2]^{1/2} . \quad (10)$$

The system has been reduced to first-order equations. Notice that $z'' \leq 0$ always. It follows that if $z' > 0$, then z will reach a maximum and then the geodesic reverses direction.

Integrating the z equation, we have

$$\int_{z_i}^z \frac{dz}{[(\alpha + E_i^2) - e^{2z}W_i^2]^{1/2}} = \pm(\eta - \eta_i) . \quad (11)$$

This completes the solution. Let the initial time be $\eta_i = 0$. Firstly, if $W_i^2 = 0$, the trajectories are simple lines,

$$W_i^2 = 0 \quad z = z_i \pm \gamma\eta, \quad (12)$$

with $\gamma = (\alpha + E_i^2)^{1/2}$. If $W_i \neq 0$ we can integrate (11) and then (2) to obtain

$$\begin{aligned} W_i^2 e^{2z} &= \gamma^2 \frac{1}{\cosh^2(\gamma\eta \mp \beta_i)}, \\ x &= x_i + \frac{\Pi_{xi}}{W_i^2} \gamma [\tanh(\gamma\eta \mp \beta_i) \pm \tanh(\beta_i)], \\ y &= y_i + \frac{\Pi_{yi}}{W_i^2} \gamma [\tanh(\gamma\eta \mp \beta_i) \pm \tanh(\beta_i)]. \end{aligned} \quad (13)$$

The last constant of integration that appears here is

$$\tanh \beta_i = \sqrt{\gamma^2 - e^{2z_i}W_i^2} / \gamma . \quad (14)$$

Notice that, unless a particle is shot directly along the axis, it will never reach $z = +\infty$. The constant, which is proportional to $\tanh(\beta_i)$, could be absorbed into x_i , but then x_i and would not correspond to the value $x(\eta_i = 0)$. Rewriting these in another way, that is useful when tracing geodesics, gives

$$\begin{aligned} e^{-z} &= e^{-z_i} \left[\gamma \cosh(\gamma\eta) \mp \sinh(\gamma\eta) \sqrt{\gamma^2 - e^{2z_i}W_i^2} \right] \\ x &= x_i + \Pi_{xi} e^{2z_i} \gamma^2 \left[\frac{\tanh(\gamma\eta)}{\gamma \mp \tanh(\gamma\eta) \sqrt{\gamma^2 - e^{2z_i}W_i^2}} \right] \\ y &= y_i + \Pi_{yi} e^{2z_i} \gamma^2 \left[\frac{\tanh(\gamma\eta)}{\gamma \mp \tanh(\gamma\eta) \sqrt{\gamma^2 - e^{2z_i}W_i^2}} \right] \end{aligned} \quad (15)$$

where

$$\frac{1}{\cosh^2 \beta_i W_i^2} = \exp(2z_i) . \quad (16)$$

We can write the path parametrically as

$$e^{2z} = \frac{\gamma^2}{W_i^2} - (x - \hat{x}_i)^2 - (y - \hat{y}_i)^2 \quad (17)$$

where $\hat{x}_i = x_i \pm (\Pi_{xi}/W_i^2)\gamma \tanh(\beta_i)$ and \hat{y}_i is defined analogously.

These trajectories have some very odd properties. As already mentioned, the trajectories never reach $z = +\infty$. Any photon travelling along increasing z eventually hits a maximum and then wraps back. As it moves toward $z = -\infty$, the velocities along x and y fall (eqns (7)) so that they never reach infinite values of x or y .

A. Geodesic flows in an expanding universe

When the expansion of space is included, the full metric becomes that of the open Friedmann universe,

$$ds^2 = -dt^2 + a^2(t) [dz^2 + e^{-2z}(dx^2 + dy^2)] . \quad (18)$$

The *null* geodesic equations are

$$\ddot{t} + H\dot{t}^2 = 0 \quad (19)$$

$$\ddot{z} + e^{-2z} (\dot{x}^2 + \dot{y}^2) + 2\mathcal{H}\dot{z} = 0 \quad (20)$$

$$\ddot{x} - 2\dot{z}\dot{x} + 2\mathcal{H}\dot{x} = 0 \quad (21)$$

$$\ddot{y} - 2\dot{z}\dot{y} + 2\mathcal{H}\dot{y} = 0 \quad , \quad (22)$$

where an overdot denotes $d/d\lambda$ and $\mathcal{H} = H\dot{t} = d \ln a / d\lambda$. Notice from (19) that

$$\dot{t} = \frac{1}{a} \quad . \quad (23)$$

so $d\lambda = a dt$.

All of the results of the previous section can quickly be adapted to the case with expansion if a time coordinate is chosen astutely. Now let $\iota = d/d\eta$. Then the geodesic equations become

$$\dot{\eta}^2 [z'' + e^{-2z} (x'^2 + y'^2)] + z' [\ddot{\eta} + 2\mathcal{H}\dot{\eta}] = 0 \quad (24)$$

$$\dot{\eta}^2 [x'' - 2y'x'] + x' [\ddot{\eta} + 2\mathcal{H}\dot{\eta}] = 0 \quad (25)$$

$$\dot{\eta}^2 [y'' - 2z'y'] + y' [\ddot{\eta} + 2\mathcal{H}\dot{\eta}] = 0 \quad . \quad (26)$$

If we choose the coordinate η such that

$$\ddot{\eta} + 2\mathcal{H}\dot{\eta} = 0, \quad (27)$$

then it follows that the geodesic equations become

$$\begin{aligned} z'' + e^{-2z} (x'^2 + y'^2) &= 0 \\ x'' - 2z'x' &= 0 \\ y'' - 2z'y' &= 0 \end{aligned} \quad (28)$$

which is precisely the same as those on a static, negatively curved hypersurface. The solutions are then the same as equations (13) (or equivalently (15)) with $\gamma = 1$ for photons and

$$\eta = \int \frac{d\lambda}{a^2} = \int \frac{dt}{a(t)} \quad (29)$$

is the usual conformal time.

For timelike geodesics, the motion can again be projected onto a static hypersurface with a new affine parameter. Again, we recover the geodesic flows of the previous section with $\alpha = -1$ except the time parameter for massive particles is not conformal time but rather [8]

$$\eta = \int \frac{dt}{a(v_o + a^2)^{1/2}} \quad (30)$$

and $v_o = av(t)/\sqrt{1-v^2}$ and $v^2(t) = g_{ij}\dot{x}^i\dot{x}^j$.

III. TRACING GEODESICS

The microwave background provides a sensitive probe of the curvature of the universe. If we locate the Earth (or our near-Earth satellite) at the origin of the coordinate system, then only photons from the surface of last scatter which travel along radial geodesics will be observed. The geodesic equations are then greatly simplified with respect to the direction of observation on the sky.

The photons seen in the sky today can be traced backwards to locate their point of origin. We have six unknowns:

$$\begin{aligned} z_i &\leq \geq \beta_i & z'_i \\ x_i & & \Pi_{xi} \\ y_i & & \Pi_{yi} \end{aligned} \quad (31)$$

The magnitude of z'_i is fixed by eqn (10) once the other initial coordinates are specified. The choice z'_i corresponds then to a choice of the sign in eqn (10). There are two sets of boundary conditions: (i) The photon position vector is at the origin, and (ii) The velocity vector is opposite to the unit vector pointing in the direction of observation. In other words,

$$\begin{aligned} (i) \quad \vec{x} &= \vec{x}_o = \vec{0} \\ (ii) \quad \hat{v} &= -\hat{n}(\theta, \phi) \end{aligned} \quad (32)$$

Boundary condition (i) places the Earth at the origin of the coordinate system. Using the geodesic solutions, we have

$$\begin{aligned} W_i^2 &= \frac{1}{\cosh^2(\eta_o \mp \beta_i)}, \\ x_i &= -\frac{\Pi_{xi}}{W_i^2} [\tanh(\eta_o \mp \beta_i) \pm \tanh(\beta_i)], \\ y_i &= -\frac{\Pi_{yi}}{W_i^2} [\tanh(\eta_o \mp \beta_i) \pm \tanh(\beta_i)] \end{aligned} \quad (33)$$

When evaluated at the origin, the geodesic equations relate the components of the velocity vector today to their initial values:

$$\begin{aligned} z'_o &= 1 - W_i^2 \\ x'_o &= \Pi_{xi} \\ y'_o &= \Pi_{yi} \end{aligned} \quad (34)$$

This velocity vector is normalized to 1 as it must be for photons. Using

$$\hat{n} = \hat{r} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} \quad , \quad (35)$$

we can rotate this into the (x, y, z) coordinate system at the origin to find

$$\begin{aligned} \hat{z} &\leq \geq \hat{k} \\ \hat{x} &\leq \geq \hat{j} \\ \hat{y} &\leq \geq \hat{i} \end{aligned} \quad (36)$$

Condition (ii) then gives

$$\begin{aligned} z'_o &= -\cos\theta \\ x'_o &= \Pi_{xi} = -\sin\theta \cos\phi \\ y'_o &= \Pi_{yi} = -\sin\theta \sin\phi \end{aligned} \quad (37)$$

> It follows that $W_i^2 = \sin^2\theta$. Also, taking the derivative of the geodesic equation,

$$z'_o = -\tanh(\eta_o \mp \beta_i) = -\cos\theta, \quad (38)$$

from which it follows that

$$\mp \tanh(\beta_i) = \frac{\cos\theta - \tanh(\eta_o)}{1 - \tanh(\eta_o) \cos\theta} \quad (39)$$

$\beta_i = \pm(\eta_o - \operatorname{arctanh}(\cos\theta))$. Putting (37) into (33) we find

$$\begin{aligned} e^{-z_i} &= \cosh(\eta_o) - \sinh(\eta_o) \cos\theta \\ e^{-z_i} x_i &= \sin\theta \cos\phi \sinh(\eta_o) \\ e^{-z_i} y_i &= \sin\theta \sin\phi \sinh(\eta_o) \end{aligned} \quad (40)$$

These are all radial geodesics in (r, θ, ϕ) ; that is, $r = \eta_o - \eta$. The integrated Sachs-Wolfe effect considers only radial geodesics; but processes such as gravitational lensing, that can deflect a photon into the line of sight, would draw from the more general pool of non-radially directed photons.

IV. LOCAL AND GLOBAL ANISOTROPY

The full geodesics on a universe of negative curvature have been obtained in an explicit form most accessible to cosmologists. One arena of renewed interest where the full geodesics may be needed is the case of a small universe. Negatively curved spacetimes can be made small and finite through topological identifications. A small universe could be witnessed with periodic effects or by features in the power spectrum of the microwave background. Topology induces global anisotropy even when the underlying space is locally isotropic. Local anisotropy is also possible in the absence of topological identifications. When homogeneous anisotropies are present, either in the form of shear or rotation in the expansion of the universe, or possibly also in the three-curvature of space, there are only a finite number of homogeneous spaces which can provide an exact description of the geometry of space. These anisotropic spaces were first classified by Bianchi [9]. They were introduced into cosmology by Taub [10], and presented in the most efficient manner by Ellis and MacCallum [11]. Since the microwave sky is currently the most significant historical record of the primordial radiation, it is instructive to show how the anisotropic sky patterns created by these different anisotropic universes can be predicted just from

a knowledge of the group invariances that generate the homogeneous geometries and their geodesic flows.

In order to determine the detailed sky patterns permitted by the Bianchi geometries it is necessary to solve for the evolution of the geodesics on the anisotropic cosmological models either exactly or approximately (in the case of small anisotropy); see for example refs. [12,13,14,16,17]. Again, the most unusual features arise in open universes. The basic quadrupole pattern arises in the simplest (flat) Bianchi type I universe with zero curvature. The addition of negative curvature focuses this quadrupole into a small hotspot on the sky (there is a preferred direction because there is a direction of lowest and highest expansion rate) in type V universes, which still possess isotropic 3-curvature. If anisotropic curvature is added then we reach the most general class of anisotropic homogeneous spaces and a spiralling of the geodesics is added to the quadrupole or focused quadrupole in the flat or open type VII universes. These particular models have been studied in the past by linearizing the geodesic equations about the isotropic solutions in which the temperature anisotropy of the microwave background is zero [12,13,14,15,16,17,18,19]. They describe the most general anisotropic distortions of flat and open Friedmann universes. However, it is also possible to predict the geometric sky patterns expected in the different Bianchi type universes by simply noting the nature of the groups of motions which define each homogeneous space.

The Bianchi classification of spatially homogeneous anisotropic universes is based on the geometric classification of 3-parameter Lie groups. The action of these groups on the spacelike hypersurfaces of constant time in these universes can be prescribed by three transformations of cartesian coordinates (x, y, z) . Each model possesses two simple translations in the $x-y$ plane, with generators $\partial/\partial x$ and $\partial/\partial y$, together with a more complicated motion out of this plane which is different for each group type. If it is considered as a flow from the $z = 0$ plane to some other plane, $z = \alpha = \text{constant}$, then the nature of this flow tells us qualitatively what the microwave background anisotropy pattern will look like. In the simplest flat universe of Bianchi type I the z -flow is uniform and just maps $(x, y) \rightarrow (x, y)$. This corresponds to a pure quadrupole geodesic temperature anisotropy pattern. In the open Bianchi type V universe the z -flow is a pure dilation and maps $(x, y) \rightarrow e^\alpha(x, y)$, with α constant. This dilation describes the hotspot created by the focusing of the quadrupole pattern in open anisotropic universes. The most general non-compact homogeneous universes containing the Friedmann models are of Bianchi type VII_h, which contain the flat Friedmann universes when $h = 0$ and the open Friedmann universes when $h \neq 0$. In type VII_h the z -flow is a rotation plus a dilation: circles of radius $r = (x^2 + y^2)^{1/2} = 1$ are mapped into circles of radius $r = e^{\alpha\sqrt{h}}$ and rotated by a constant angle α . Ob-

servationally, this corresponds to the geodesics producing a focusing of the basic quadrupole (as in type V) with a superimposed spiral twist. In Bianchi type VII₀ there is simply a spiral added to the underlying quadrupole with no focusing because the 3-geometry is flat. The z -flow for the closed universes of Bianchi type IX, which contain the closed Friedmann models as special cases, is more complicated. The z -plane action corresponds to the following $SO(3)$ invariant motion in polar coordinates based on the x -axis, in the region $r < 2\pi$, [20]:

$$\{r, \theta, \phi\} = \{\cos^{-1}[(1 - \beta^2)^{1/2} \cos(\alpha/2)], \\ \sin^{-1} \beta[1 - (1 - \beta^2) \cos^2(\alpha/2)]^{-1/2}, \frac{1}{2}\alpha + \phi_0\}$$

Sky patterns in other, less familiar, Bianchi types can be generated in a similar fashion, if required. Thus, type VI_h is generated by a z -flow that combines a shear with a dilation: the hyperbolae $x^2 - y^2 = A^2$ are mapped into hyperbolae which are rotated by a hyperbolic angle β into $x^2 - y^2 = A^2 e^{2\beta\sqrt{-h}}$.

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